

FURTHER TRIGONOMETRY

Text

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Further Trigonometry

1 Radian Measure

You will be familiar with using degrees to measure angles, but this is not the only way to do it. In fact, in some areas of mathematics it is not even the most convenient way.

Radians are an alternative unit of measure for angles, often used in trigonometry.

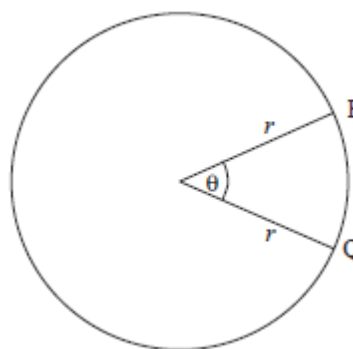
Radians relate the arc length of a sector to its angle.



Worked Example 1

The figure opposite shows a circle of radius r and arc PQ which is subtended by an angle of θ degrees.

- What is the length of the complete circumference?
- What is the arc length PQ in terms of θ and r ?
- If $PQ = r$, what is the value of θ ?



Solution

- Circumference of circle = $2\pi r$

- Arc length PQ = $\frac{\theta}{360} \times 2\pi r$

- If $PQ = r$,

$$\text{then } r = \frac{\theta}{360} \times 2\pi r$$

$$\text{then } \theta = \frac{360r}{2\pi r}$$

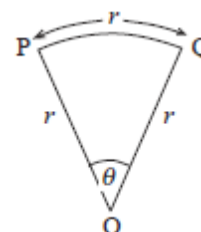
$$\text{then } \theta = \frac{180}{\pi} \text{ degrees}$$

The Worked Example above gives the clue as to how radians are defined.

One **radian** corresponds to the angle which gives the same arc length as the radius. So if θ is the angle POQ in degrees,

$$\text{Then, when } PQ = r, \quad \theta = \frac{180}{\pi} \text{ degrees (see above)}$$

$$\text{So } 1 \text{ radian} = \frac{180}{\pi} \text{ degrees.}$$



1

This is not a very convenient definition, so it is usual to note that:

π radians $\equiv$ 180 degrees

Note also that:

$$1 \text{ degree} \equiv \frac{\pi}{180} \text{ radians}$$



Worked Example 2

Convert the following angles in degrees to radians.

- (a) 90° (b) 360° (c) 720° (d) 60°



Solution

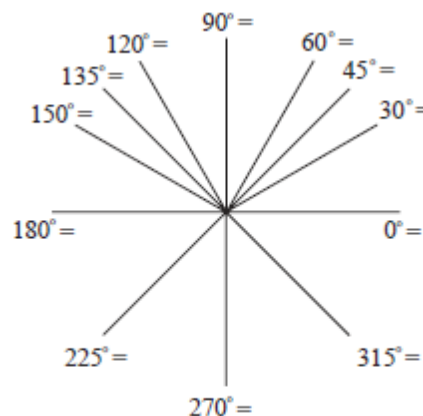
Since $180^\circ = \pi$ radians,

- (a) $90^\circ = \frac{\pi}{2}$ radians (b) $360^\circ = 2\pi$ radians
 (c) $720^\circ = 4\pi$ radians (d) $60^\circ = \frac{\pi}{3}$ radians



Worked Example 3

Copy the diagram opposite, putting in the equivalent radian measure for each angle given in degrees.



Solution

Working anticlockwise from 0° , the conversions are as follows:

- (i) $0^\circ = 0 \times \frac{\pi}{180} \text{ radians} = 0 \text{ radians}$
 (ii) $30^\circ = 30 \times \frac{\pi}{180} \text{ radians} = \frac{\pi}{6} \text{ radians}$
 (iii) $45^\circ = 45 \times \frac{\pi}{180} \text{ radians} = \frac{\pi}{4} \text{ radians}$
 (iv) $60^\circ = 60 \times \frac{\pi}{180} \text{ radians} = \frac{\pi}{3} \text{ radians}$
 (v) $90^\circ = 90 \times \frac{\pi}{180} \text{ radians} = \frac{\pi}{2} \text{ radians}$
 (vi) $120^\circ = 120 \times \frac{\pi}{180} \text{ radians} = \frac{2\pi}{3} \text{ radians}$
 (vii) $135^\circ = 135 \times \frac{\pi}{180} \text{ radians} = \frac{3\pi}{4} \text{ radians}$
 (viii) $150^\circ = 150 \times \frac{\pi}{180} \text{ radians} = \frac{5\pi}{6} \text{ radians}$

$$(ix) \quad 180^\circ = 180 \times \frac{\pi}{180} \text{ radians} = \pi \text{ radians}$$

$$(x) \quad 225^\circ = 225 \times \frac{\pi}{180} \text{ radians} = \frac{5\pi}{4} \text{ radians}$$

$$(xi) \quad 270^\circ = 270 \times \frac{\pi}{180} \text{ radians} = \frac{3\pi}{2} \text{ radians}$$

$$(xii) \quad 315^\circ = 315 \times \frac{\pi}{180} \text{ radians} = \frac{7\pi}{4} \text{ radians}$$



Worked Example 4

As you have seen, there is a one to one correspondence between degrees and radians so that

$$\theta^\circ = \theta \times \frac{\pi}{180} \text{ radians or } \theta \text{ radians} \equiv \left(\theta \times \frac{180}{\pi} \right)^\circ$$

Use the formulae above to convert:

$$(a) \quad \text{to radians:} \quad (i) \quad 75^\circ \quad (ii) \quad 20^\circ \quad (iii) \quad 105^\circ$$

$$(b) \quad \text{to degrees:} \quad (i) \quad \frac{2\pi}{3} \text{ radians} \quad (ii) \quad \frac{\pi}{12} \text{ radians}$$



Solution

$$(a) \quad (i) \quad 75^\circ = 75 \times \frac{\pi}{180} \text{ radians} = \frac{5\pi}{12} \text{ radians}$$

$$(ii) \quad 20^\circ = 20 \times \frac{\pi}{180} \text{ radians} = \frac{\pi}{9} \text{ radians}$$

$$(iii) \quad 105^\circ = 105 \times \frac{\pi}{180} \text{ radians} = \frac{7\pi}{12} \text{ radians}$$

$$(b) \quad (i) \quad \frac{2\pi}{3} \text{ radians} = \left(\frac{2\pi}{3} \times \frac{180}{\pi} \right)^\circ = 120^\circ$$

$$(ii) \quad \frac{\pi}{12} \text{ radians} = \left(\frac{\pi}{12} \times \frac{180}{\pi} \right)^\circ = 15^\circ$$



Exercises

1. Convert these angles in radians to degrees:

$$(a) \quad \frac{7\pi}{6} \text{ radians} \quad (b) \quad 3\pi \text{ radians} \quad (c) \quad 2 \text{ radians} \quad (d) \quad \frac{11\pi}{12} \text{ radians}$$

2. Convert these angles in degrees to radians.

$$(a) \quad 12\frac{1}{2}^\circ \quad (b) \quad 72\frac{1}{2}^\circ \quad (c) \quad 210^\circ \quad (d) \quad 20^\circ$$

2 Arc and Sector Area

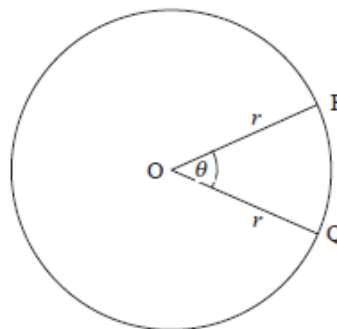
You have already have seen how to find the arc length.

If θ is measured in degrees, then

$$\text{arc length PQ} = \frac{\theta}{360} \times 2\pi r = \frac{\pi r \theta}{180}$$

but if θ is measured in radians,

$$\text{arc length PQ} = \frac{\theta}{2\pi} \times 2\pi r = r\theta.$$



The region between the two radii and the arc is called a *sector* of the circle. There is a simple formula for the area of a sector of a circle subtended by an angle.

$$\text{Area of sector} = \frac{\theta}{360} \times \pi r^2$$

but if θ is measured in radians,

$$\text{area of sector} = \frac{\theta}{2\pi} \times \pi r^2$$

$$\text{area of sector} = \frac{1}{2} r^2 \theta$$

So, using radian measure, the formulae for both arc length and sector area take a simple form, namely

$$\text{arc length} = r\theta$$

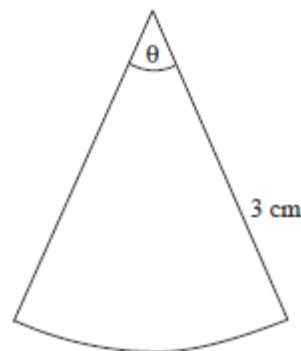
$$\text{sector area} = \frac{1}{2} r^2 \theta$$

You will see how these formulae are used in the following example.



Worked Example 1

- A silver pendant is made in the form of a sector of a circle as shown opposite. If the radius is 3 cm, what is the angle, θ , in radians so that the area is 6 cm^2 ?
- Another pendant has the same total perimeter, but with radius 2.5 cm. What is the required angle, θ , in radians?



Solution

$$(a) \text{ Sector area} = 6 = \frac{1}{2} \times 3^2 \times \theta$$

$$\Rightarrow \theta = \frac{4}{3} \text{ radians}$$

2

$$\begin{aligned} \text{(b) Total perimeter length} &= 3 + 3 + 3 \times \frac{4}{3} \\ &= 10 \text{ cm} \end{aligned}$$

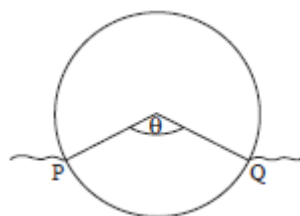
Hence, for the second pendant,

$$\begin{aligned} 10 &= 2.5 + 2.5 + 2.5 \times \theta \\ \Rightarrow \theta &= 2 \text{ radians} \end{aligned}$$



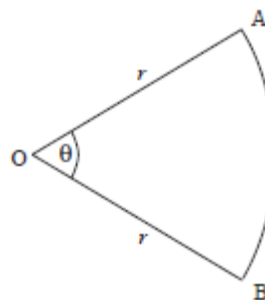
Exercises

1. An oil drum of diameter 60 cm is floating as shown in the diagram on the right.
 - (a) Given that the arc length PQ is 50 cm, find θ in radians.
 - (b) The drum is 1 metre long. Find the volume of the drum which lies below the surface level.



2. In the diagram on the right, the length of the arc AB is 20 cm and the area of the sector AOB is 100 cm^2 .

Form two equations involving r and θ when θ is measured in radians. Solve these equations simultaneously to find r and θ .



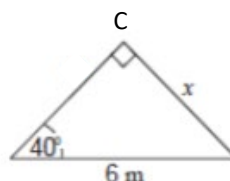
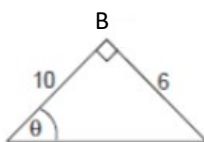
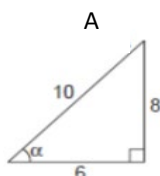
3. A stone, swung round on the end of a 200 cm string, completes an arc subtending an angle at the centre of 2 radians every second. Find the speed at which the stone is moving.

3 Sine, Cosine and Tangent Functions

You are probably familiar with the uses of sine, cosine and tangents in right angled triangles, but the following activities will further revise the concepts.



Worked Example 1



- (a) Write down the sine, cosine and tangent of the angle marked in triangle A.
- (b) Find the angle marked, to 1 d.p, in triangle B.
- (c) Find the length marked x to 2 d.p. in triangle C.



Solution

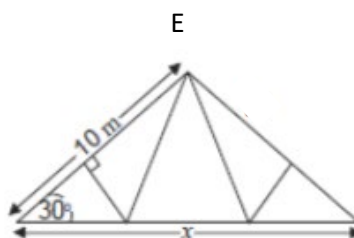
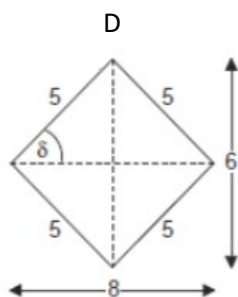
$$(a) \sin \theta = \frac{8}{10} \quad \cos \theta = \frac{6}{10} \quad \tan \theta = \frac{8}{6}$$

$$(b) \tan \theta = \frac{6}{10} \quad \theta = \tan^{-1} \frac{6}{10} \quad \theta = 31.0^\circ \text{ to 1 d.p.}$$

$$(c) \sin 40 = \frac{x}{6} \quad 6 \sin 40 = x \quad x = 3.86 \text{ to 2 d.p.}$$



Worked Example 2



(a) Find the angle δ in the rhombus labelled D.

(b) The sketch labelled E shows the frame of a roof. Find the width x .



Solution

$$(a) \sin \delta = \frac{3}{5} \quad \delta = \sin^{-1} \frac{3}{5} \quad \delta = 36.9^\circ \text{ to 1 d.p.}$$

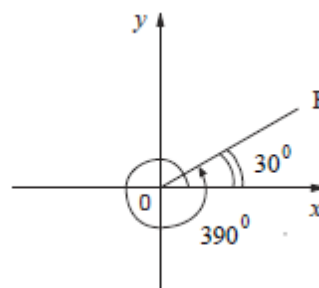
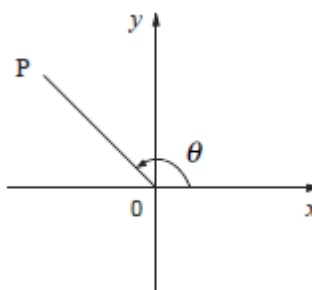
$$(b) \cos 30 = \frac{x/2}{10} \quad 10 \cos 30 = \frac{x}{2} \quad 10 \frac{\sqrt{3}}{2} = \frac{x}{2} \quad x = 10\sqrt{3}$$

You have seen how sine, cosine and tangent are defined for angles between 0° and 90° but this can be extended to other angles.

Angles of more than 90° can be defined as the angle θ made between a rotating 'arm' OP and the positive x axis, as shown opposite. It is possible to define angles of more than 360° in this way, or even negative angles, as shown in the next diagram.

If the length of OP is 1 unit, then the sine, cosine and tangent of any angle is defined in terms of the x and y co-ordinates of the point P as follows;

$$\sin \theta = y \quad \cos \theta = x \quad \tan \theta = \frac{y}{x}$$

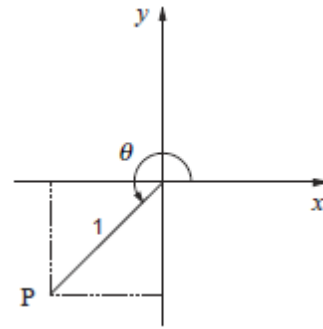




Note

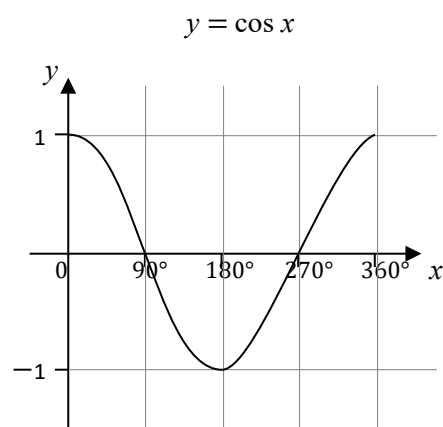
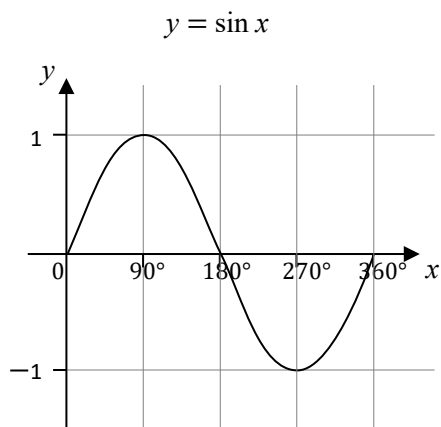
$\sin \theta$, $\cos \theta$ and $\tan \theta$ may be negative for certain values of θ . For instance in the figure opposite, y and x are negative, whilst $\tan \theta$ is positive.

Scientific calculators give sine, cosine and tangent of any angle.



The functions $y = \sin x$ and $y = \cos x$ can be plotted on a graphic calculator or computer, or indeed by hand, for every 10° interval between 0° and 360° . A smooth curve is drawn through the points.

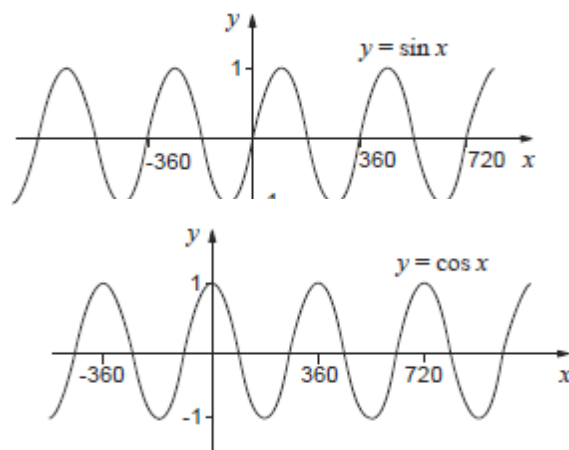
The maximum and minimum values of the function y are 1 and -1 .



Both functions, $\sin x$ and $\cos x$, lie between ± 1 . The amplitude is the height from the x -axis (or centre-line) to the peak (or trough). The amplitude for both sine and cosine functions is 1.

Whilst we have only plotted the functions for $0 \leq x \leq 360^\circ$, it is easy to see how it can be extended.

For the range $360^\circ \leq x \leq 720^\circ$, the pattern will repeat itself, and similarly for negative values. A period is the length of one cycle, before the pattern repeats. The period for both sine and cosine functions is 360° . The graphs are illustrated below.



You can see from these graphs that $y = \sin x$ is an **odd** function

($y(-x) = -y(x)$), whilst $y = \cos x$ is an **even** function ($y(-x) = y(x)$).

You can always use your calculator to find values of sine and cosine of an angle although sometimes it is easier to use the properties of the functions.



Worked Example 3

Show that $\sin 30^\circ = \frac{1}{2}$ and find the following without using a calculator:

- (a) $\sin 150^\circ$ (b) $\sin (-30^\circ)$ (c) $\sin 210^\circ$
 (d) $\sin 390^\circ$ (e) $\cos 60^\circ$ (f) $\cos 120^\circ$
 (g) $\cos (-60^\circ)$ (h) $\cos 240^\circ$



Solution

From the sketch opposite of an equilateral triangle,

$$\sin 30^\circ = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

(a) $\sin 150^\circ = \sin (180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2}$

(b) $\sin (-30^\circ) = -\sin 30^\circ$ (function is odd)
 $= -\frac{1}{2}$

(c) $\sin 210^\circ = -\frac{1}{2}$

(d) $\sin 390^\circ = \sin 30^\circ = \frac{1}{2}$

(e) Since $\cos \theta = \sin (90^\circ - \theta^\circ)$

$$\cos 60^\circ = \sin (90^\circ - 60^\circ)$$

$$= \sin 30^\circ$$

$$= \frac{1}{2}$$

(f) $\cos 120^\circ = -\cos 60^\circ$ (function is odd about 90°)

$$= -\frac{1}{2}$$

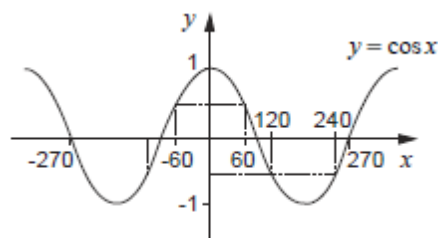
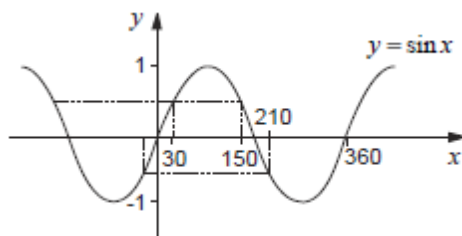
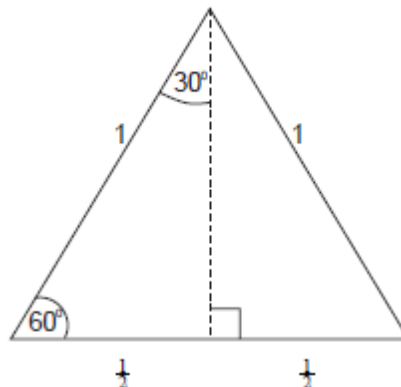
(g) $\cos (-60^\circ) = \cos 60^\circ = \frac{1}{2}$ (function is even)

(h) $\cos 240^\circ = \cos 120^\circ = -\frac{1}{2}$

In the example above, the relationship

$$\cos \theta = \sin (90^\circ - \theta^\circ)$$

was used.



Similarly, it should be noted that

$$\sin \theta = \cos (90 - \theta^\circ)$$

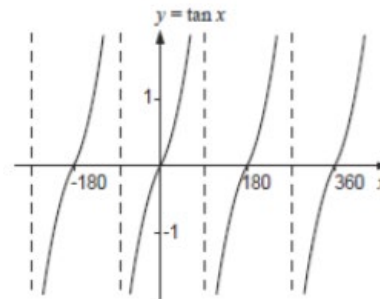
The last function to consider here is $\tan x$, which has rather different properties from $\sin x$ and $\cos x$.

We can use a graphic calculator or computer to sketch the function

$$y = \tan x$$

for x in any range. The period of the function is the distance between any two consecutive asymptotes; the period for $y = \tan x$ is 180° .

Unlike the sine and cosine functions, $\tan x$ is not bounded by ± 1 . In fact, as x increases to 90° , $\tan x$ increases without limit.



Worked Example 4

Using the values

$$\tan 45^\circ = 1 \quad \text{and} \quad \tan 30^\circ = \frac{1}{\sqrt{3}}$$

find, without using a calculator,

- (a) $\tan 135^\circ$ (b) $\tan (-45^\circ)$ (c) $\tan 315^\circ$
 (d) $\tan (-30^\circ)$ (e) $\tan 150^\circ$ (f) $\tan 60^\circ$

Solution

(a) $\tan 135^\circ = \tan (180^\circ - 45^\circ) = -\tan 45^\circ = -1$

(b) $\tan (-45^\circ) = -\tan 45^\circ$
 $= -1$

(c) $\tan 315^\circ = -\tan 45^\circ$
 $= -1$

(d) $\tan (-30^\circ) = -\tan 30^\circ = -\frac{1}{\sqrt{3}}$

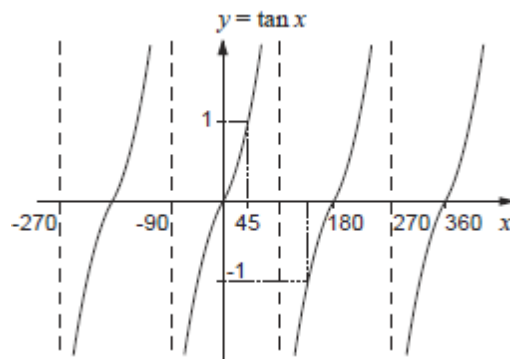
(e) $\tan 150^\circ = -\tan 30^\circ = -\frac{1}{\sqrt{3}}$

(f) $\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \frac{\cos 30^\circ}{\sin 30^\circ} = \frac{1}{\tan 30^\circ}$
 $= \frac{1}{\left(\frac{1}{\sqrt{3}}\right)} = \sqrt{3}$

In (f), the result is equivalent to using

$$\tan \theta = \frac{1}{(\tan 90 - \theta)}$$

This is a useful property for the exercises below.



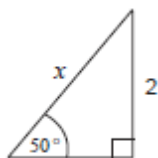
Further properties are summarised in section 5 .



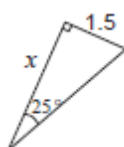
Exercises

1. Find the length marked x to 2 d.p. for each of the figures below.

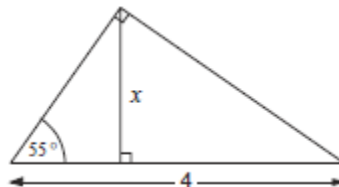
(a)



(b)



(c)



2. Given that $\sin 45^\circ = \frac{1}{\sqrt{2}}$, find, without using a calculator,

(a) $\sin 135^\circ$

(b) $\sin (-45^\circ)$

(c) $\sin 315^\circ$

(d) $\cos 45^\circ$

(e) $\cos (225^\circ)$

(f) $\tan 45^\circ$

(g) $\tan 135^\circ$

(h) $\tan (-45^\circ)$

3. Prove that $\sin \theta = \cos(90 - \theta)$.

4 Solving Trigonometrical Equations

Trigonometric functions can be used to model physical phenomena but applying these functions to problems in the real world will often result in a trigonometric equation to solve. In this section, you will consider some simple equations to solve. These will highlight the difficulties that can occur.



Worked Example 1

Solve $\sin x = \frac{1}{2}$ for $0^\circ \leq x \leq 360^\circ$.



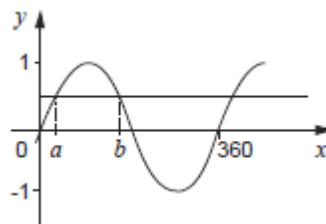
Solution

In solving equations of this sort it is **vital** to be aware that there may be **more than one** possible solution in the allowable domain - this possibility results from the periodic nature of this function. It is usually helpful to make a sketch of the relevant function, and this will help to identify the number of possible solutions.

In this case, the functions to plot are

$$y = \frac{1}{2} \text{ and } y = \sin x$$

The points of intersection in the range $0^\circ \leq x \leq 360^\circ$ are solutions of the equation.



The figure shows that there are just **two** solutions, denoted by a and b , in the range. You can find the value of a by entering 0.5 in your calculator and using the 'inverse' sin or 'arc sin' buttons.

This gives $a = 30^\circ$ (or $\frac{\pi}{6} \approx 0.524$ radians, if you are using radians).

Your calculator will only give you this single value but it is easy to find b provide you have used a sketch. This shows that the function $y = \sin x$ is **symmetrical** about $x = 90^\circ$ and so $b = 90^\circ + 60^\circ = 150^\circ$ (≈ 2.618 radians). So the equation has two solutions, namely 30° or 150° .



Note

Even though your calculator cannot directly give you both solutions, it can be usefully used to check the answers. To do this enter 30° and press the $\boxed{\sin}$ button; similarly, for 150° .

If in the example above, the range had been -360° to 360° , there would be two further solutions. Using a sketch, you may like to consider what they would be. Again, you can check these solutions using the $\boxed{\sin}$ button on your calculator.



Worked Example 2

Solve $3 \cos x = -0.6$ for $0^\circ \leq x \leq 360^\circ$, giving your answer to 1 d.p.



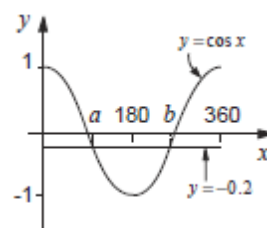
Solution

Rearranging the equation

$$\cos x = -\frac{0.6}{3} = -0.2$$

As before, you plot the functions

$$y = \cos x \text{ and } y = -0.2$$



There are two solutions, a and b , with a between 90° and 180° and b between 180° and 270° . Enter -0.2 into your calculator and use the 'inverse' cosine button - this will give you the answer

$$a = 101.5^\circ$$

From the graph, and using symmetry, the second solution, b , is given by

$$b = 180^\circ + 180^\circ - a \approx 78.5^\circ$$

$$\Rightarrow b = 258.5^\circ$$

and to 1 d.p. the solutions are 101.5° and 258.5° .

(Remember to check the second solutions on your calculator.)

Challenge:

What would be the solutions to the previous problem if the range had been given as $-360^\circ \leq x \leq 0^\circ$?

**Worked Example 3**

Solve the equation $\tan x = -2$ where x is measured in radians and $-2\pi \leq x \leq 2\pi$. Give your answers to 2 d.p. (Set your calculator to radian mode.)

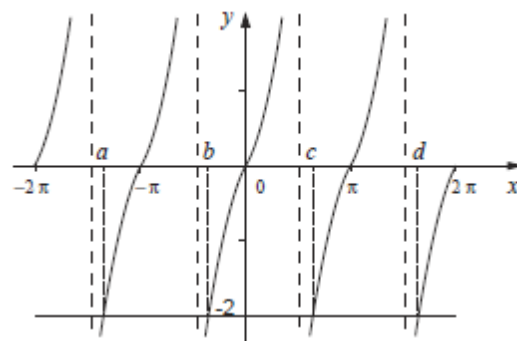
**Solution**

Note that x is given in radians rather than degrees here and remember that π radians $\equiv 180^\circ$.

The graph of $y = \tan x$ and $y = -2$ is shown opposite.

You can see that there are four solutions denoted by a, b, c and d .

To find one of these solutions, enter -2 in your calculator and use the 'inverse tan' (or 'arctan') button. This should give -1.107 radians, so that $b = -1.107$.



Since the function $\tan x$ has period π (or 180°),

$$a = b - \pi = -4.249$$

$$b \text{ given above}$$

$$c = b + \pi = 2.035$$

$$d = b + 2\pi = 5.177$$

To 2 decimal places the solutions in radians are

$$-4.25, -1.11, 2.04 \text{ and } 5.18$$

**Exercises**

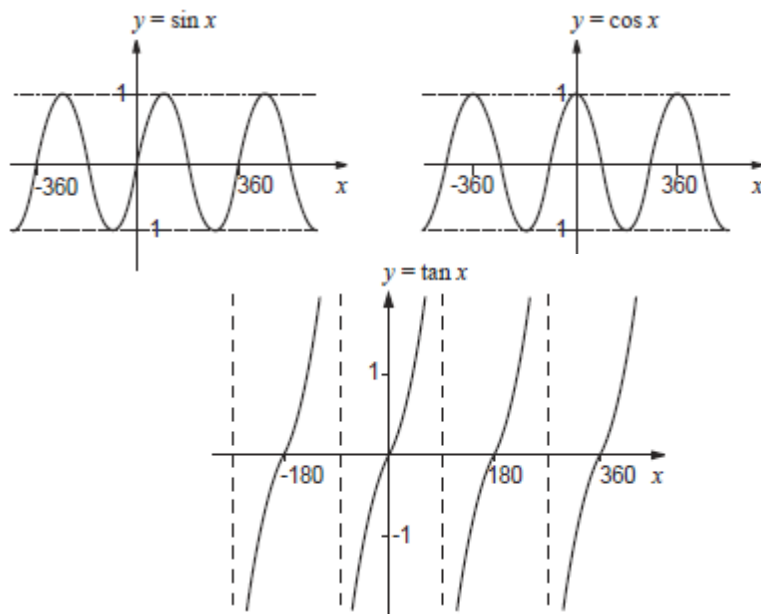
Give all answers to 3 significant figures.

1. Solve $\cos x = 0.5$, $0^\circ \leq x \leq 360^\circ$
2. Solve $\tan x = 1$, $0^\circ \leq x \leq 360^\circ$
3. Solve $\sin x = \frac{1}{4}$, $0^\circ \leq x \leq 180^\circ$
4. Solve $\sin x = -0.5$, $0^\circ \leq x \leq 360^\circ$
5. Solve $4 \tan x = 1$, $0^\circ \leq x \leq 720^\circ$
6. Solve $\sin x = -\frac{1}{3}$, $-\pi \leq x \leq \pi$

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7. Solve $3 \cos x = 1 - \pi \leq x \leq \pi$
8. Which of these equations have no solutions?
- (a) $\sin x = 1, 0^\circ \leq x \leq 360^\circ$
 - (b) $\cos x = -\frac{1}{4}, 0^\circ \leq x \leq 90^\circ$
 - (c) $\cos x = 2, 0^\circ \leq x \leq 360^\circ$
 - (d) $\tan x = 2, 0^\circ \leq x \leq 90^\circ$
 - (e) $4 \sin x = -5, 0^\circ \leq x \leq 360^\circ$

5 Properties of Trig Functions



There are a number of important properties, which you have already been using. These are summarised below:

<u>Degrees</u>	<u>Radians</u>
1. $\cos(x) = \sin(90^\circ - x)$	$\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$
2. $\sin(x) = \cos(90^\circ - x)$	$\sin(x) = \cos\left(\frac{\pi}{2} - x\right)$
3. $\cos(180^\circ - x) = -\cos(x)$	$\cos(\pi - x) = -\cos(x)$
4. $\sin(180^\circ - x) = +\sin(x)$	$\sin(\pi - x) = \sin(x)$
5. $\tan x = \frac{\sin x}{\cos x}$	
6. $\sin^2 x + \cos^2 x = 1$	

The last result is a very important one and is proved below for $0 < x < 90^\circ$.

Theorem $\sin^2 x + \cos^2 x = 1$

Proof

Using Pythagoras' Theorem,

$$BC^2 + AB^2 = AC^2$$

Dividing by AC^2 ,

$$\left(\frac{BC}{AC}\right)^2 + \left(\frac{AB}{AC}\right)^2 = 1.$$

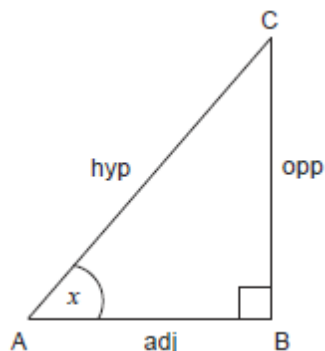
But

$$\sin x = \frac{\text{opp}}{\text{hyp}} = \frac{BC}{AC}$$

$$\cos x = \frac{\text{adj}}{\text{hyp}} = \frac{AB}{AC}$$

giving

$$\sin^2 x + \cos^2 x = 1$$



Note

$\sin^2 x$ is used for $(\sin x)^2$ so that there is no confusion between $(\sin x)^2$ and $\sin(x^2)$. These properties can be used to solve more complex trig equations.

Worked Example 1

Find all solutions between 0° and 360° of the equation

$$\sin x = 2 \cos x \text{ to 2 d.p.}$$

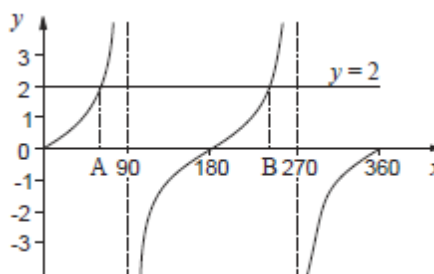
Solution

Assuming $\cos x \neq 0$, you can divide both sides of the equation by $\cos x$ to give

$$\tan x = 2$$

(since $\frac{\sin x}{\cos x} = \tan x$).

The graph of $y = \tan x$ and $y = 2$ is shown opposite. This shows that there are 2 solutions in the range 0° to 360° .



Using a calculator, the inverse tangent of 2 is 63.4° (to 1 d.p.). This is marked A on the graph. Since the tangent graph has a period of 180° , B is given by

$$63.4^\circ + 180^\circ = 243.4^\circ.$$

Note that at the start of the solution, $\cos x = 0$ was excluded.

When $\cos x = 0$, this gives $\sin x = 0$. These two equations are not both true for any value of x .

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Worked Example 2

Find all solutions between 0° and 360° of the equation

$$\cos x \sin x = 3 \cos x$$

giving answers to 1 d.p.



Solution

The equation can be rewritten as

$$\cos x \sin x - 3 \cos x = 0$$

$$\text{or} \quad \cos x (\sin x - 3) = 0$$

$$\text{So either } \cos x = 0 \text{ or } \sin x - 3 = 0 \Rightarrow \sin x = 3$$

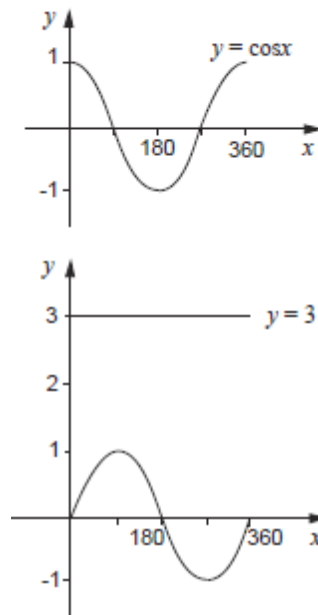
The graph of $y = \cos x$ is shown opposite for $0^\circ \leq x \leq 360^\circ$. The solutions are given by $\cos x = 0$, which gives

$$x = 90^\circ \text{ or } 270^\circ$$

The other possibility is $\sin x = 3$. The graph of $y = \sin x$ and $y = 3$ are shown opposite.

They do not intersect so there are no solutions.

The only solutions of the equation are therefore $x = 90^\circ$ or 270° .



The example above showed that algebraic techniques like factorising can be applied to equations involving trigonometric functions. The next example shows that **quadratics** in sines or cosines can also be solved.



Worked Example 3

Find all solutions between 0° and 360° to the equation

$$3 \cos x = 2 \sin^2 x$$



Solution

This equation contains a mixture of 'cos' and 'sin' terms, but using the identity

$$\cos^2 x + \sin^2 x = 1$$

will make it possible to express the equation in 'cos' terms only.

$$\text{Now} \quad \sin^2 x = 1 - \cos^2 x$$

so that the equation becomes

$$\begin{aligned} 3 \cos x &= 2(1 - \cos^2 x) \\ &= 2 - 2 \cos^2 x \end{aligned}$$

Hence $2 \cos^2 x + 3 \cos x - 2 = 0$.

(This is a quadratic equation in $\cos x$.)

This can be factorised to give

$$(2 \cos x - 1)(\cos x + 2) = 0$$

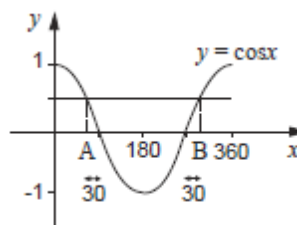
and either $2 \cos x - 1 = 0$ or $\cos x + 2 = 0$

The first equation gives

$$\cos x = \frac{1}{2}$$

and the graphs of $y = \cos x$ and $y = \frac{1}{2}$

are shown in the figure opposite



Using the inverse cosine function on a calculator gives $x = 60^\circ$ and this corresponds to A on the figure. Using symmetry about 180° gives B as

$$270^\circ + 30^\circ = 300^\circ$$

The second equation, $\cos x = -2$, has no solutions. The only possible solutions of the original equation are 60° and 300° .



Exercises

- Find all the solutions to the equation $\sin x \cos x + \sin x = 0$ for $0^\circ \leq x \leq 360^\circ$
- Solve completely the equation $1 - \sin x = 2 \cos^2 x$, $0^\circ \leq x \leq 360^\circ$. (Hint: there are three solutions.)
- Find all solutions between 0° and 360° to 1 d.p., of all the equations $5 - 2 \cos x = 8 \sin^2 x$.
- Find all the solutions between 0 and 2π radians of the equation $\sin x = -\cos x$, leaving π in your answers.
- Solve $3 \sin x = \cos x$ for $0^\circ \leq x \leq 360^\circ$.