

FURTHER CALCULUS

Text

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Further Calculus

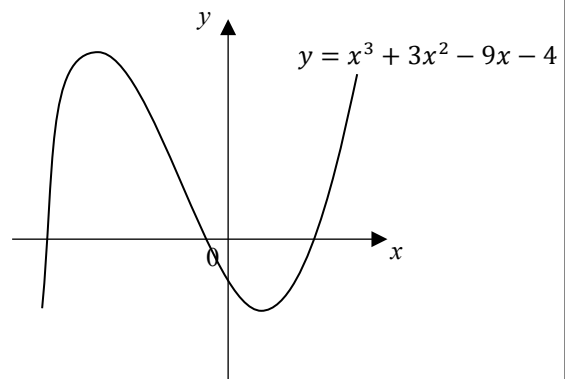
Introduction

You should have already covered the earlier material introducing Calculus before starting this Unit so, by now, you will already have met the idea of both differentiation and integration, and you will have used the techniques developed in the earlier Units to solve problems. This section extends the range of problems you can solve, including finding the greatest or least value of a function and differentiating complicated functions.

1 Rate of Change of the Gradient

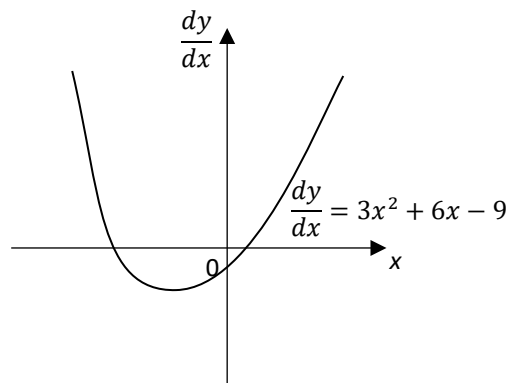
The sketch opposite shows the graph of

$$y = x^3 + 3x^2 - 9x - 4$$



You have already seen that the derivative of this function is given by

$$\frac{dy}{dx} = 3x^2 + 6x - 9$$

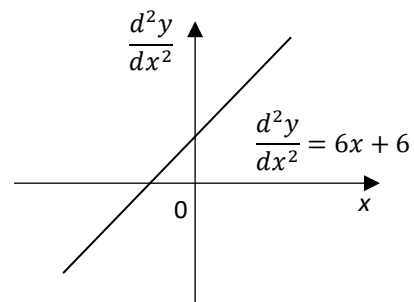


This is also illustrated opposite.

The gradient function, $\frac{dy}{dx}$, is also a function of x and can be differentiated again to give the **second** differential

$$\frac{d^2y}{dx^2} = 6x + 6$$

Again, this is illustrated opposite.





Exercises

1. Find the second derivative of these functions:

- | | | |
|------------------|-----------------------|----------------------------|
| (a) $y = x^3$ | (b) $y = x^4$ | (c) $y = x^2$ |
| (d) $y = x$ | (e) $y = \frac{1}{x}$ | (f) $y = 4x^3 - 12x^2 + 5$ |
| (g) $y = 3x + 1$ | (h) $y = e^x$ | (i) $y = \ln x$ |

2. Find $\frac{d^3y}{dx^3}$ if y is

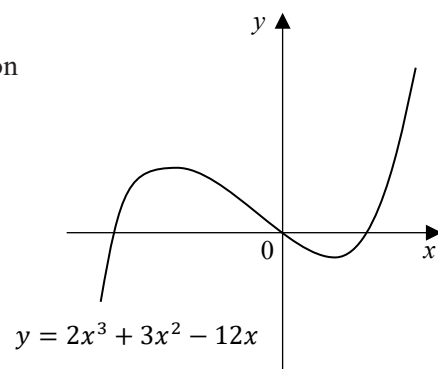
- | | | |
|-------------|----------------------------|-----------|
| (a) x^4 | (b) $5x^2 + \frac{3}{x^2}$ | (c) e^x |
| (d) $\ln x$ | (e) $2x$ | |

2 Stationary Points

In this section you will consider the curve with equation

$$y = 2x^3 + 3x^2 - 12x$$

This is cubic, and a rough sketch of its graph is shown opposite. It has two **stationary points** (sometimes called **turning points**) at which the gradient is zero.



We now need to find the coordinates of the stationary points.

For example, we will see how to find the coordinates of the stationary points for the curve with equation

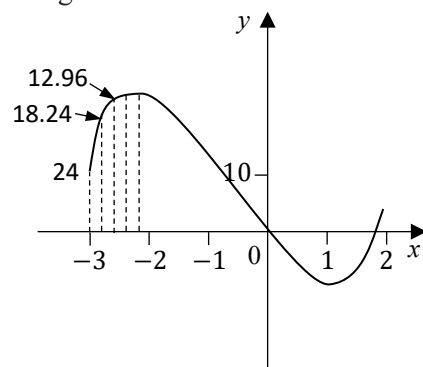
$$y = 2x^3 + 3x^2 - 12x$$

In the *Rates of Change* Unit, you saw that the nature of the stationary points can be determined by looking at the gradient on each side of the stationary point. Here an alternative, more formal method is developed, based on using second derivatives.

Drawing an accurate sketch of the curve with equation

$$y = 2x^3 + 3x^2 - 12x$$

between $x = -3$ and $+2$, you can see the change in the value of the gradient as it rapidly decreases until it reaches zero at the point of maximum value.



For a **maximum** value of a function, note that the gradient is decreasing in value as it passes through the value zero at the stationary point, whereas for a **minimum** value this gradient is increasing.

The result can be summarised as

For **stationary points** of a function $y(x)$

$$\frac{dy}{dx} = 0$$

If $\frac{d^2y}{dx^2} < 0$ at a stationary point, it corresponds to a **maximum** value of y .

If $\frac{d^2y}{dx^2} > 0$ at a stationary point, it corresponds to a **minimum** value of y .



Worked Example 1

Find maxima and minima of the curve with equation

$$y = -\frac{1}{4}x^4 + \frac{1}{3}x^3 - 6x^2 + 3$$

Hence sketch the curve.



Solution

For stationary points, $\frac{dy}{dx} = 0$, which gives

$$\frac{dy}{dx} = x^3 + x^2 - 12x$$

So $\frac{dy}{dx} = 0$

$$\Rightarrow x^3 + x^2 - 12x = 0$$

$$x(x^2 + x - 12) = 0$$

$$x(x + 4)(x - 3) = 0$$

Hence there are stationary points at $x = 0$, -4 and 3 . To find out their nature, the second derivative is used. Now

$$\frac{d^2y}{dx^2} = 3x^2 + 2x - 12$$

At $x = 0$, $\frac{d^2y}{dx^2} = -12 < 0$

$\Rightarrow$ a maximum at $x = 0$ of value $y = 3$.

At $x = -4$, $\frac{d^2y}{dx^2} = 3(-4)^2 + 2(-4) - 12 = 28 > 0$

$\Rightarrow$ minimum at $x = -4$ of value

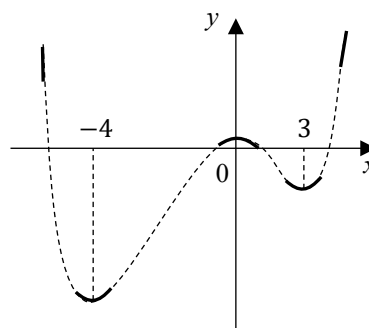
$$y = \frac{1}{4}(-4)^4 + \frac{1}{3}(-4)^3 - 6(-4)^2 + 3 = -\frac{151}{3}$$

At $x = 3$, $\frac{d^2y}{dx^2} = 3(3)^2 + 2(3) - 12 = 21 > 0$

$\Rightarrow$ minimum at $x = 3$ of value

$$y = \frac{1}{4}(3)^4 + \frac{1}{3}(3)^3 - 6(3)^2 + 3 = -\frac{87}{4}$$

The information so far found can be sketched on a graph (not to scale). You also know that as $x \rightarrow \pm\infty, y \rightarrow +\infty$. It is now clear how to sketch the shape, shown dashed on the diagram.



There is one further type of stationary point to be considered and that is a **point of inflection**. This is considered in the next example.



Worked Example 2

Find the stationary point of $y = x^3$. What is the value of $\frac{d^2y}{dx^2}$ at the stationary point? Sketch the graph of $y = x^3$.



Solution

If $y = x^3$, then $\frac{dy}{dx} = 3x^2$ and $\frac{dy}{dx} = 0 \Rightarrow x = 0$.

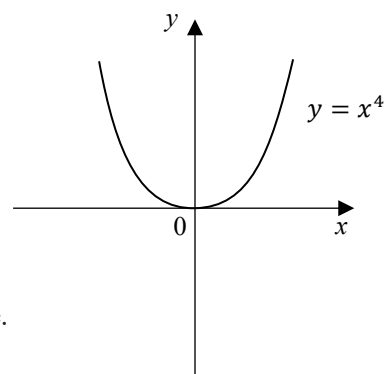
So $x = 0$ is a stationary point and so we look at the second derivative to test whether it is a maximum or minimum; here

$$\frac{d^2y}{dx^2} = 6x = 0 \text{ at } x = 0$$

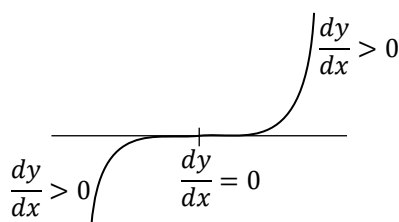
Note that $\frac{d^3y}{dx^3} = 6$ at $x = 0$ and so $\frac{d^3y}{dx^3} \neq 0$

So this is not a maximum or a minimum but at $x = 0$ there is a point of inflection.

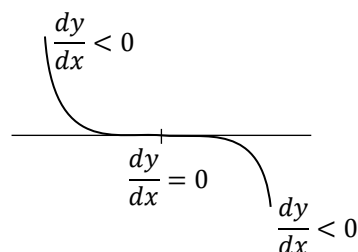
We can also see that as $x \rightarrow +\infty$ then $y \rightarrow +\infty$ and as $x \rightarrow -\infty$ then $y \rightarrow -\infty$. We can now sketch the curve.



For a **horizontal** point of inflection, not only does $\frac{dy}{dx} = 0$, but also $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$ at the point.



These are **sufficient** but not **necessary** conditions, as can be seen by considering $y = x^5$.



Worked Example 3

Find the stationary point of $y = x^5$.

What is the value of $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ at its stationary point?

Sketch the graph of $y = x^5$.



Solution

If $y = x^5$, then $\frac{dy}{dx} = 5x^4 = 0 \Rightarrow x = 0$.

Hence we have a stationary point at $x = 0$.

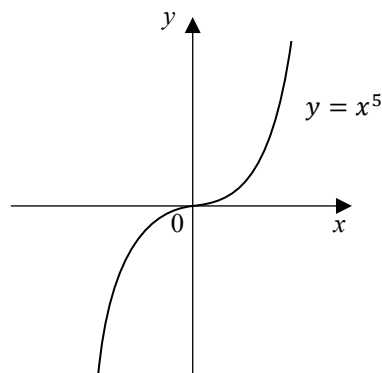
Now $\frac{d^2y}{dx^2} = 20x^3$
 $= 0$ at $x = 0$;

and $\frac{d^3y}{dx^3} = 60x^2$
 $= 0$ at $x = 0$.

Note though that $\frac{d^4y}{dx^4} = 120x$
 $= 0$ at $x = 0$

and $\frac{d^5y}{dx^5} = 120 > 0$ when $x = 0$

The sketch of $y = x^5$ is shown opposite, with its point of inflection at $x = 0$.



Worked Example 4

Find the nature of the stationary points of the curve with equation

$$y = x^4 + 4x^3 - 6$$

Sketch a graph of the curve.

2



Solution

Now

$$\begin{aligned}\frac{dy}{dx} &= 4x^3 + 12x^2 \\ &= 0 \text{ when } 4x^3 + 12x^2 = 0\end{aligned}$$

$$\Rightarrow 4x^2(x + 3) = 0$$

$$\Rightarrow x = 0, -3 \text{ for stationary points.}$$

But $\frac{d^2y}{dx^2} = 12x^2 + 24x$

At $x = 0$, $\frac{d^2y}{dx^2}$, but $\frac{d^3y}{dx^3} = 24x + 24 = 24 > 0$ at $x = 0$.

So there is a point of inflection at $(0, -6)$.

At $x = -3$, $\frac{d^2y}{dx^2} = 12(-3)^2 + 24(-3)$
 $= 108 - 72 = 36 > 0$

So there is a minimum at $x = -3$ of value

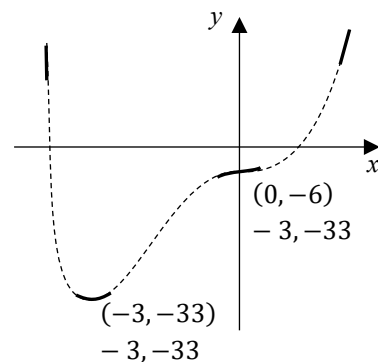
$$\begin{aligned}y(-3) &= (-3)^4 + 4(-3)^3 - 6 \\ &= 81 - 4 \times 27 - 6 \\ &= -33\end{aligned}$$

To sketch the curve, also note that

$$y \rightarrow \infty \text{ as } x \rightarrow \pm\infty$$

You can then deduce the form of the curve, shown dashed opposite.

Finally in this section, it should be noted that the conditions given earlier are **sufficient** conditions to guarantee max/min values, but not **necessary**.



Worked Example 5

Draw the graph of $y = x^4$.

Does it have any maximum or minimum values?

Do the conditions hold?



Solution

Here is a sketch of the graph of $y = x^4$.

It clearly has one minimum value at $x = 0$.



Now if $y = x^4$, then $\frac{dy}{dx} = 4x^3$
 $= 0$ at $x = 0$

and $\frac{d^2y}{dx^2} = 12x^2$
 $= 0$ at $x = 0$

Continuing to higher derivatives:

$$\frac{d^3y}{dx^3} = 24x$$

$$= 0 \text{ at } x = 0$$

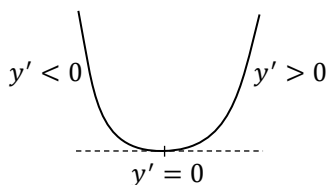
but $\frac{d^4y}{dx^4} = 24 > 0$ at $x = 0$

So the original conditions do not hold.

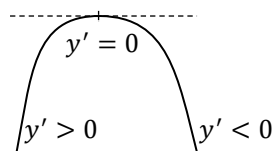
By now you could be getting confused with the rules to determine the nature of stationary points. So let it be stressed that:

- (a) Stationary points are always given by $\frac{dy}{dx} = 0$.
- (b) If, at the stationary point, $\frac{d^2y}{dx^2} > 0$ there is a minimum, whereas if $\frac{d^2y}{dx^2} < 0$ there is a maximum.
- (c) If, at the stationary point, $\frac{d^2y}{dx^2} = 0$, then there is a point of inflexion provided $\frac{d^3y}{dx^3} \neq 0$.

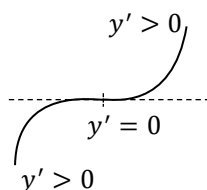
Conditions (b) and (c) are sufficient to guarantee the nature of the stationary point but, as you have already seen, **not** necessary. When this analysis does not hold, that is, when $\frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = 0$ at a stationary point, it is easier to consider the **sign** of the gradient each side of the stationary point.



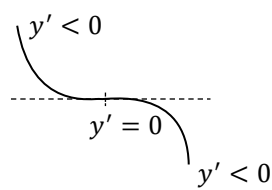
Gradients $< : 0 : >$
 minimum



$> : 0 : <$
 maximum



Gradients $> : 0 : >$
 point of inflexion



$< : 0 : <$
 point of inflexion

2



Worked Example 6

Show that $y = x^4$ has a minimum at $x = 0$ by checking on the values of the derivative on each side of the stationary point.

Solution

$$\begin{aligned}\text{For stationary points, } \frac{dy}{dx} &= 4x^3 \\ &= 0 \Rightarrow x = 0\end{aligned}$$

$$\text{But, if } x = 0, \quad \frac{d^2y}{dx^2} = 12x^2 = 0$$

$$\text{and} \quad \frac{d^3y}{dx^3} = 24x = 0$$

So you cannot use the usual results, but at, for example,

$$x = -0.1 \Rightarrow \frac{dy}{dx} = 4(-0.1)^3 = -0.004 < 0$$

$$x = 0.1 \Rightarrow \frac{dy}{dx} = 4(0.1)^3 = 0.004 > 0$$

So there really is a **minimum** at $x = 0$.



Exercises

- Find the turning points of these curves using differentiation. In each case, find out whether the points found are maxima, minima or points of inflection.

(a) $y = 2x^2 + 3x - 1$

(b) $y = x^3 - 12x + 6$

(c) $y = x^3 + 2x^2 - 5x - 6$

(d) $y = x^2 - 4x + 2 \ln(x)$

(e) $y = e^x - 4x$

(f) $y = e^x + 5$

- Investigate the nature of the stationary points of the curve

$$y = (x - 1)(x - a)^2$$

when (a) $a > 1$, (b) $0 < a < 1$. In each case give a sketch of the curve.

3 Differentiating Composite Functions

The crucial result needed here is the ‘function of a function’ rule. It can be justified by noting that if y is a function of u , and u is a function of x ,

that is,

$$y = f(u) \text{ and } u = g(x)$$

then a small change, δx say, in x will result in a small change, say δu say in u , which in turn results in a small change, δy in y . Now

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$$

and if you let $\delta x \rightarrow 0$, then $\delta u \rightarrow 0$ and $\delta y \rightarrow 0$ and the equation becomes

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

For example, suppose

$$y = (3x + 2)^2$$

then one way to differentiate this function is to multiply out the brackets and differentiate term by term;

$$\begin{aligned} y &= (3x + 2)(3x + 2) \\ &= 9x^2 + 12x + 2 \end{aligned}$$

and
$$\frac{dy}{dx} = 18x + 12$$

Another method, which will be even more useful as the functions get more complicated, is to introduce a new variable u defined by

$$u = 3x + 2$$

so that
$$y = u^2$$

Now $\frac{dy}{du} = 2u$ and $\frac{du}{dx} = 3$, so using the result above,

$$\frac{dy}{dx} = (2u) \times 3 = 6(3x + 2), \text{ as before}$$



Worked Example 1

Use both methods described above to differentiate the function $y = (3 - 2x)^3$



Solution

We introduce the new variable $u = 3 - 2x$ so that $y = u^3$.

Hence $\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = -2$, giving

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
 &= 3u^2 \times (-2) \\
 &= -6u^2 \\
 &= -6(3 - 2x)^2
 \end{aligned}$$

The alternative method is to multiply out the bracket cubed to give:

$$\begin{aligned}
 y &= 27 - 54x + 36x^2 - 8x^3 \\
 \text{giving } \frac{dy}{dx} &= -54 + 72x - 24x^2 \\
 &= -6(9 - 12x + 4x^2) \\
 &= -6(3 - 2x)^2
 \end{aligned}$$

as before.



Worked Example 2

If $y = e^{x^2}$, find $\frac{dy}{dx}$.



Solution

As before, introduce a new variable u defined by

$$u = x^2$$

so that

$$y = e^u$$

Now $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 2x$

giving $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (e^u)2x = 2xe^{x^2}$

Note: reversing the result in the Example above gives $\int xe^{x^2} dx = \frac{1}{2}e^{x^2} + c$ (constant).

Another important result needed is given by

$$\boxed{\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}}$$

but note that this is only true for **first** derivatives.

For example, if $y = x^2$

then $\frac{dy}{dx} = 2x$

But, expressing x as a function of y ,

$$x = y^{\frac{1}{2}}$$

and $\frac{dx}{dy} = \frac{1}{2}y^{-\frac{1}{2}} = \frac{1}{2y^{\frac{1}{2}}} = \frac{1}{2x}$ (generalising the result that $\frac{d}{dx}(x^n) = nx^{n-1}$ for any n)

So, as expected $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$



Worked Example 3

Verify the result $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ when $y = \frac{1}{2}(x + 3)$



Solution

Here $\frac{dy}{dx} = \frac{1}{2}$ and expressing x in terms of y gives:

$$x = 2y - 3$$

and $\frac{dx}{dy} = 2$; this verifies the result $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

The proof of this important result is based on using small increments (increases) δx and δy , noting that

$$\frac{\delta y}{\delta x} = \frac{1}{\frac{\delta x}{\delta y}}$$

and taking the limit as $\delta x \rightarrow 0$ (and $\delta y \rightarrow 0$).

This section is concluded by using this result in finding the derivative of the function

$$y = a^x$$

To achieve this you must first take 'logs' of both sides to give

$$\ln y = \ln(a^x) = x \ln a \text{ (properties of logs)}$$

Defining $u = (\ln a)x \Rightarrow \frac{du}{dx} = \ln a$

and $u = \ln y$

Hence $\frac{du}{dy} = \frac{1}{y}$

giving $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$

$$= \left(\frac{1}{\frac{du}{dy}} \right) \ln a$$

$$= y \ln a$$

$$\frac{d}{dx}(a^x) = (\ln a)a^x$$

so, for example,

$$\frac{d}{dx}(2^x) = (\ln 2)2^x$$

Note: if $a = e$ the formula above becomes $\frac{d}{dx}(e^x) = (\ln e)e^x$ but $\ln e = 1$ so we have the important result that:

$$\frac{d}{dx}(e^x) = e^x$$



Exercises

1. Differentiate these functions:

(a) $y = (2x - 5)^8$ ($u = 2x - 5$)

(b) $y = (x^2 + x^3)^{10}$ ($u = x^2 + x^3$)

(c) $y = \frac{1}{x-2}$ ($u = x - 2, y = u^{-1}$)

(d) $y = \frac{1}{(3x+1)^2}$ ($u = 3x + 1, y = u^{-2}$)

(e) $y = \sqrt{x+1}$ ($u = x + 1, y = u^{\frac{1}{2}}$)

(f) $y = (e^x - x)^6$ ($u = e^x - x$)

(g) $y = \ln(3x + 4)$ ($u = 3x + 4$)

(h) $y = e^{\sqrt{x}}$ ($u = \sqrt{x}$)

2. Find any turning points on these curves:

(a) $y = e^{x^2}$

(b) $y = e^{-(x+4)^2}$ ($u = -(x + 4)^2$)

(c) $y = (\ln x)^2$ ($u = \ln(x)$)

(d) $y = \left(x + \frac{1}{x}\right)^4$ ($u = x + \frac{1}{x}$)

3. A kettle contains water which is cooling according to the equation

$W = 80e^{-0.04t} + 20$ where W is the temperature of the water in $^{\circ}\text{C}$ at time t (in minutes) after the kettle was switched off.

Find the rate at which the water is cooling in $^{\circ}\text{C}/\text{min}$ when $t = 30$ and when $t = 60$.

4. The population of a country is modelled by the function

$$P = 12 \times (1.03)^t$$

where P is the population in millions and t is the time in years after the start of 1990.

Find the rate at which the population is increasing in millions per year at the start of the year 2000.

5. (a) Differentiate e^{2x} . Hence write down $\int e^{2x} dx$.
 (b) Differentiate e^{-x} , and so write down $\int e^{-x} dx$.

6. Differentiate e^{x^2} , and use your result to find $\int xe^{x^2} dx$.

7. The derivative of $\ln x$ is $\frac{1}{x}$.

By differentiating a suitable logarithmic function, find $\int \frac{1}{x+2} dx$.

4 Integration Again

The results that have been developed in the last section are, as you will see, very useful in integration. For example, if

$$y = (x + 5)^4$$

then $\frac{dy}{dx} = 4(x + 5)^3$ (using the methods in the last section).

Hence $\int 4(x + 5)^3 dx = (x + 5)^4 + C$ (C is arbitrary constant)

or $\int (x + 5)^3 dx = \frac{1}{4}(x + 5)^4 + C'$ ($C' = \frac{1}{4}C$)



Worked Example 1

Differentiate $y = (ax + b)^{n+1}$ and hence deduce the value of

$$\int (ax + b)^n dx$$

Use your results to find

- (a) $\int (3x - 1)^9 dx$ (b) $\int \frac{1}{(2x+4)^2} dx$ (c) $\int \sqrt{x-1} dx$



Solution

If $y = (ax + b)^{n+1}$, then we know that:

$$\frac{dy}{dx} = a(n+1)(ax + b)^n$$

and integrating both sides gives

$$\int a(n+1)(ax+b)^n dx = (ax+b)^{n+1} + c \text{ (constant)}$$

$$a(n+1) \int (ax+b)^n dx = (ax+b)^{n+1} + c \text{ (constant)}$$

This gives: $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + k \text{ (constant)}$

(a) Here $n = 9$, $a = 3$ and $b = -1$ giving:

$$\begin{aligned} \int (3x-1)^9 dx &= \frac{(3x-1)^{9+1}}{3(9+1)} + k \text{ (constant)} \\ &= \frac{(3x-1)^{10}}{30} + k \end{aligned}$$

(b) Here $n = -2$, $a = 2$ and $b = 4$ Type equation here. giving:

$$\begin{aligned} \int (2x+4)^{-2} dx &= \frac{(2x+4)^{-2+1}}{2(-1)} + k \text{ (constant)} \\ &= \frac{(2x+4)^{-1}}{-2} + k \\ &= -\frac{1}{2(2x+4)} + k \end{aligned}$$

(c) Here $n = \frac{1}{2}$, $a = 1$ and $b = -1$ giving:

$$\begin{aligned} \int \sqrt{x-1} dx &= \int (x-1)^{\frac{1}{2}} dx \\ &= \frac{(x-1)^{\frac{1}{2}+1}}{1(\frac{1}{2}+1)} + k \\ &= \frac{2}{3} (x-1)^{\frac{3}{2}} + k \end{aligned}$$

Similarly, it can be seen that if

$$y = \ln(ax+b)$$

Then $\frac{dy}{dx} = \frac{a}{(ax+b)}$

Hence $\int \frac{adx}{(ax+b)} = \ln(ax+b) + C$

or

$$\int \frac{dx}{(ax+b)} = \frac{1}{a} \ln(ax+b) + C'$$

$$C' = \frac{1}{a} C$$



Worked Example 2

Use the result above to evaluate

(a) $\int \frac{dx}{(x+1)}$

(b) $\int \frac{dx}{1-2x}$

4



Solution

$$(a) \int \frac{dx}{(x+1)} = \ln(x+1) + c \text{ (constant)}$$

$$(b) \int \frac{dx}{(1-2x)} = \frac{1}{(-2)} \ln(1-2x) + c \text{ (constant)}$$

$$= -\frac{1}{2} \ln(1-2x) + c \text{ (constant)}$$

Another important method of integration is based on the result that if

$$y = \ln f(x)$$

then defining $u = f(x)$, $y = \ln u$ and

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{u} \times f(x) \\ &= \frac{f'(x)}{f(x)} \end{aligned}$$

Hence

$$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + C$$



Worked Example 3

Find $\int \frac{(3x^2+x)}{(2x^3+x^2)} dx$



Solution

If $f(x) = 2x^3 + x^2$, then $f'(x) = 6x^2 + 2x$, so the integral can be written as

$$\begin{aligned} I &= \int \frac{(3x^2+x)}{(2x^3+x^2)} dx \\ &= \frac{1}{2} \int \frac{2(3x^2+x)}{(2x^3+x^2)} dx \\ &= \frac{1}{2} \int \frac{f'(x)}{f(x)} dx \quad (\text{when } f(x) = 2x^3 + x^2) \\ &= \frac{1}{2} \int \ln f(x) + C \\ &= \frac{1}{2} \ln(6x^2 + 2x) + C \end{aligned}$$



Exercises

1. Find

(a) $\int (2x + 1)^4 dx$

(b) $\int \frac{1}{(x-5)^2} dx$

(c) $\int \frac{1}{\sqrt{x+1}} dx$

(d) $\int \sqrt{4x-1} dx$

2. Evaluate

(a) $\int_0^1 \frac{dx}{(x+1)}$

(b) $\int_1^2 \frac{dx}{(2x-1)}$

3. Find

(a) $\int \frac{dx}{(3x+2)}$

(b) $\int \frac{x}{x^2+1} dx$

(c) $\int \frac{e^x}{1+e^x} dx$

4. Evaluate $\int_0^1 x e^{-x^2} dx$

5 Differentiating Products and Quotients

These are very useful formulae for differentiating both products and quotients. For example, if y is defined as the product of the functions, u and v of x , then

$$y(x) = u(x)v(x)$$

Using the basic definition of a derivative, let δx be a small change in x , then consider

$$\begin{aligned} y(x + \delta x) - y(x) &= u(x + \delta x)v(x + \delta x) - u(x)v(x) \\ &= [u(x + \delta x) - u(x)]v(x + \delta x) + u(x)[v(x + \delta x) - v(x)] \\ &\quad \text{(the middle two terms cancel)} \end{aligned}$$

Dividing both sides by δx gives

$$\frac{y(x + \delta x) - y(x)}{\delta x} = \frac{(u(x + \delta x) - u(x))}{\delta x} v(x + \delta x) + u(x) \frac{(v(x + \delta x) - v(x))}{\delta x}$$

and taking the limit as $\delta x \rightarrow 0$, gives

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

(since $v(x + \delta x) \rightarrow v(x)$ as $\delta x \rightarrow 0$)

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Worked Example 1

If $y = x^2(x - 1)^2$, find $\frac{dy}{dx}$ by

- (a) using the formula above,
- (b) multiplying out and differentiating term by term.



Solution

- (a) Here $u = x^2$, $v = (x - 1)^2$,

$$\begin{aligned}\text{So } \frac{dy}{dx} &= 2x(x - 1)^2 + x^2 2(x - 1) \\ &= 2x(x - 1)(x - 1 + x) \\ &= 2x(x - 1)(2x - 1)\end{aligned}$$

- (b) $y = x^2(x^2 - 2x + 1) = x^4 - 2x^3 + x^2$

$$\begin{aligned}\frac{dy}{dx} &= 4x^3 - 6x^2 + 2x \\ &= 2x(2x^2 - 3x + 1) \\ &= 2x(x - 1)(2x - 1) \text{ (as before)}\end{aligned}$$



Worked Example 2

If $y = (x + 1)e^{-x}$, find $\frac{dy}{dx}$.



Solution

This time, you must use the formula, with $u = x + 1$, $v = e^{-x}$.

$$\begin{aligned}\text{So } \frac{dy}{dx} &= 1 \times (e^{-x}) + (x + 1)(-e^{-x}) \\ &= -xe^{-x}\end{aligned}$$

Turning to the equivalent formula for a quotient, with

$$y = \frac{u}{v} = u \left(\frac{1}{v} \right)$$

which is the product of u and $\frac{1}{v}$.

Now

$$\begin{aligned}\frac{d}{dx} \left(\frac{1}{v} \right) &= \frac{d}{dv} \left(\frac{1}{v} \right) \frac{dv}{dx} && \text{('function of a function')} \\ &= -\frac{1}{v^2} \frac{dv}{dx}\end{aligned}$$

and

$$\begin{aligned}\frac{dv}{dx} &= \frac{du}{dx} \times \frac{1}{v} + u \frac{d}{dx} \left(\frac{1}{v} \right) \\ &= \frac{du}{dx} \times \frac{1}{v} + u \left(-\frac{1}{v^2} \frac{dv}{dx} \right)\end{aligned}$$

That is

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{\left(v \frac{du}{dx} - u \frac{dv}{dx} \right)}{v^2}$$



Worked Example 3

Use the quotient formula to find $\frac{d}{dx}((1+x)e^{-x})$.



Solution

Here $u = (1+x)$, $v = e^x$, so that

$$y = \frac{u}{v} = \frac{1+x}{(e^x)} = (1+x)e^{-x}$$

$$\begin{aligned}\text{Thus } \frac{dy}{dx} &= \frac{e^x \times 1 - (1+x)e^x}{(e^x)^2}, \text{ since } \frac{d}{dx}(e^x) = e^x \\ &= -\frac{xe^x}{(e^x)^2} \\ &= -\frac{x}{e^x} \\ &= -xe^{-2} \quad (\text{as in the previous example})\end{aligned}$$

So you can see that the quotient formula is just another form of the product formula. It is though sometimes very convenient to use.



Worked Example 4

Differentiate $y = \frac{x-1}{2x-3}$ with respect to x .



Solution

Here $u = x-1$, $v = 2x-3$ and using the quotient formula,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1 \times (2x-3) - 2(x-1)}{(2x-3)^2} \\ &= \frac{2x-3-2x+2}{(2x-3)^2} \\ &= \frac{-1}{(2x-3)^2}\end{aligned}$$



Worked Example 5

Find any stationary points for the curve with equation

$$y = \frac{x^3}{(1+x)}.$$

Sketch the curve.



Solution

Here $u = x^3$, $v = 1 + x$, so $y = \frac{u}{v}$ and using the quotient formula

$$\begin{aligned} \frac{dy}{dx} &= \frac{3x^2(1+x) - x^3 \times 1}{(1+x)^2} \\ &= \frac{3x^2 + 2x^3}{(1+x)^2} \end{aligned}$$

For stationary points,

$$\begin{aligned} 3x^2 + 2x^3 &= 0 \\ x^2(3 + 2x) &= 0 \\ \Rightarrow x &= 0, -\frac{3}{2} \end{aligned}$$

To determine the nature of these stationary points, you can check the sign of $\frac{dy}{dx}$ each side of the points.

So, for $x = 0$,

$$\begin{aligned} \frac{dy}{dx}(-0.1) &= \frac{3(-0.1)^2 + 2(-0.1)^3}{(1-0.1)^2} = \frac{0.03 - 0.002}{(0.9)^2} \\ &= \frac{0.028}{0.81} > 0 \end{aligned}$$

$$\begin{aligned} \text{and } \frac{dy}{dx}(0.1) &= \frac{3(0.1)^2 + 2(0.1)^3}{(1+0.1)^2} = \frac{0.03 + 0.002}{(1.1)^2} \\ &= \frac{0.032}{1.21} > 0 \end{aligned}$$

So there is a point of inflection at $x = 0$ and value of function here is $y = 0$.

For $x = -1.5$,

$$\begin{aligned} \frac{dy}{dx}(-1.6) &= \frac{3(-1.6)^2 + 2(-1.6)^3}{(1-1.6)^2} = \frac{-0.512}{0.36} < 0 \\ \frac{dy}{dx}(-1.4) &= \frac{3(-1.4)^2 + 2(-1.4)^3}{(1-1.4)^2} = \frac{0.392}{0.16} > 0 \end{aligned}$$

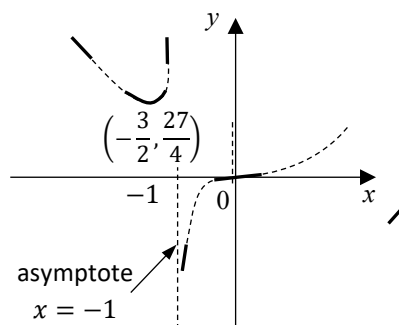
Hence there is a minimum at $x = -1.5$ of value $y = \frac{27}{4}$.

Also note that the function has an **asymptote** at $x = -1$, that is

$$y \rightarrow \pm\infty \text{ as } x \rightarrow -1$$

$$\text{As } x \rightarrow \pm\infty, y = \frac{x^3}{1+x} \approx \frac{x^3}{x} = x^2$$

All this information is marked on the diagram opposite, and the curve is sketched as a dotted line.



Exercises

1. Differentiate the following functions with respect to x .

(a) $y = xe^x$

(b) $y = x^2 \ln(x)$

(c) $y = \frac{2x+1}{x^2-3}$

(d) $y = \frac{e^x}{1+x}$

2. Find any stationary points on the curves in Question 1.

3. Find any stationary points on the curve

$$y = \frac{\ln \sqrt{x+1}}{e^x}$$

4. Differentiate $\sqrt{\frac{x+1}{2x-1}}$.

5. Differentiate $\sqrt{e^x(1+x)}$.

6. Differentiate $\frac{\ln(x-2)}{\sqrt{x}}$.

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7. Sketch the curve with equation

$$y = x^3 + \frac{3}{2}x^2 - 6x$$

by first finding all stationary points and their nature.

8. Sketch the curve with equation

$$y = \frac{1}{x^2} e^{-\frac{1}{x^2}}$$

9. Find any stationary points of the curve with equation

$$y = \frac{x^2}{1-x}$$

Hence sketch the curve, indicating any asymptotes.